

ON THE CONVERGENCE OF BEST UNIFORM DEVIATIONS

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ABSTRACT. If a function f is continuous on a closed Jordan curve Γ and meromorphic inside Γ , then the polynomials of best uniform approximation to f on Γ converge interior to Γ . Furthermore, the limit function can in each case be explicitly determined in terms of the mapping function for the interior of Γ . Applications and generalizations of this result are also given.

1. Introduction. For a continuous complex valued function f defined on a set E in the plane let $\|f\|_E = \sup_{z \in E} |f(z)|$. Also, if E is compact, for $n \in \mathbb{Z}^+$ let $p_n(f, E)$ denote the polynomial of degree n of best uniform approximation to f on E .

In general, the difference $[f(z) - p_n(f, E)(z)]$ does not converge to zero on E , and so one is naturally led to the question of whether or not this difference converges for each $z \in E$ in general. One might also inquire as to the convergence or divergence of the sequence $\{[f(z) - p_n(f, E)(z)]/\|f - p_n(f, E)\|_E\}$, which is of course a broader question.

By expanding on the work of Carathéodory and Fejér [1], Schur and Goluzin [2, p. 497], we are enabled to show that if f is continuous on Γ , a closed Jordan curve, and meromorphic in the interior of Γ , then the sequence $\{f - p_n(f, \Gamma)\}_{n=0}^\infty$ converges in the interior of Γ in the following sense. If $\{a_i\}_{i=1}^m$ are the poles of f in the interior of Γ with respective multiplicities $\{l_i\}_{i=1}^m$, then the sequence

$$\left\{ \prod_{i=1}^m \left(\frac{\phi(z) - a_i}{1 - \bar{a}_i \phi(z)} \right)^{l_i} [f(z) - p_n(f, \Gamma)(z)] \right\}_{n=0}^\infty$$

converges uniformly on compact subsets of the interior of Γ , where $w = \phi(z)$ is an analytic function that maps the interior of Γ onto the open unit disc. Furthermore, we show that the limit function for the latter sequence can be explicitly determined in terms of the function ϕ .

2. Main theorem. We will first state our main theorem in the case where Γ is the unit circle U , and then in §6 present the general case. Let D denote the interior of U . If f is as above, we can write

$$f(z) = \sigma(z) / \prod_{i=1}^m (z - a_i)^{l_i},$$

where σ is analytic in D and continuous in $D + U$, and $\sigma(a_i) \neq 0$ for $i = 1, 2, \dots, m$.

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Theorem 1. *Let f be as above; then for $z \in D$,*

$$(1) \quad \lim_{n \rightarrow \infty} [f(z) - p_n(f, U)(z)] \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} = \frac{1}{\lambda} \prod_{j=1}^K \left(\frac{z - c_j}{1 - \bar{c}_j z} \right),$$

where λ is a prescribed root of an explicitly determined polynomial, $K \leq (\sum_{i=1}^m l_i) - 1$, and the c_j 's, $j = 1, 2, \dots, K$, can be explicitly determined in terms of λ . Furthermore, the convergence is uniform on compact subsets of D .

Remark. Whether or not (1) holds for $z \in U$ is unknown. The few examples of best approximation to meromorphic functions on U that are known [5] indicate uniform convergence on U .

It should also be remarked that if f is as in Theorem 1, we can now calculate $\lim_{n \rightarrow \infty} \|f - p_n(f, U)\|_U$, and we can find (using Taylor series for instance) a sequence of polynomials $\{q_n\}_{n=0}^\infty$ such that

$$\lim_{n \rightarrow \infty} \|f - q_n\|_U = \lim_{n \rightarrow \infty} \|f - p_n(f, U)\|_U.$$

3. Preliminary results. We shall first show that if f is as in Theorem 1, then there exists a unique function g , analytic in D , which minimizes

$$\left\| [f(z) - g(z)] \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} \right\|_D,$$

or more precisely, which minimizes

$$\left\| f_1(z) - g(z) \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} \right\|_D,$$

where $f_1(z) = \sigma(z) / \prod_{i=1}^m (1 - \bar{a}_i z)^{l_i}$. We shall then show that

$$g(z) = \lim_{n \rightarrow \infty} p_n(f, U)(z) \quad \text{for } z \in D.$$

To this end we note that proving the existence and uniqueness of such a function g is equivalent to proving the existence and uniqueness of a function F analytic in D , with the prescribed values $F^{(j)}(a_i) = f^{(j)}(a_i)$, for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 1$, and which minimizes $\|F\|_D$. For if F is such a function we can write

$$(2) \quad F(z) = f_1(z) - g(z) \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i},$$

where g is analytic in D . Should there exist a function g_1 analytic in D for which

$$\left\| f_1(z) - g_1(z) \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} \right\|_D < \left\| f_1(z) - g(z) \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} \right\|_D;$$

then let

$$F_1(z) = f_1(z) - g_1(z) \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i}.$$

The function F_1 will then be analytic in D . Assume the values $F_1^{(j)}(a_i) = f_1^{(j)}(a_i)$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 1$, and $\|F_1\|_D < \|F\|_D$, thus contradicting our assumption about F . Conversely, if we start with a "minimal" g and define F by (2), we can likewise show that F has the desired minimal property.

We shall, therefore, presently concern ourselves with the problem of determining the function F . We begin by establishing its existence.

Lemma 1. *Let $\{a_i\}_{i=1}^m$ and $\{l_i\}_{i=1}^m$ be as in Theorem 1 and $\{B_i^{(j)}\}_{i=1}^m; j=0, \dots, l_i-1$ be any set of $\sum_{i=1}^m l_i$ constants. Then there exists a function F analytic in D with $F^{(j)}(a_i) = B_i^{(j)}$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 1$ which minimizes $\|F\|_D$ for all such functions.*

Proof. Clearly, the class of all such functions is nonempty. In fact, one can easily construct a polynomial P , which interpolates the given values and so since $\|P\|_D < \infty$, a straightforward application of Montel's compactness criterion [3, Vol. I, p. 415] establishes our lemma.

The following theorem completely describes the function F .

Theorem 2. *Let f_1 be as before; then there exists a unique function F analytic in D such that*

$$F^{(j)}(a_i) = f_1^{(j)}(a_i) \quad \text{for } i = 1, 2, \dots, m \text{ and } j = 0, 1, \dots, l_i - 1,$$

and such that $\|F\|_D$ is a minimum. Furthermore,

$$F(z) = \frac{1}{\lambda} \prod_{j=1}^K \left(\frac{z - c_j}{1 - \bar{c}_j z} \right),$$

where λ and the c_j 's are as in Theorem 1.

Corollary 1. *As an immediate consequence of the above theorem, we have that there exists a unique function g analytic in D which minimizes*

$$\left\| f_1(z) - g(z) \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} \right\|_D.$$

Furthermore,

$$g(z) = f(z) - \frac{1}{\lambda} \left[\prod_{i=1}^m \left(\frac{1 - \bar{a}_i z}{z - a_i} \right)^{l_i} \right] \left[\prod_{j=1}^K \left(\frac{z - c_j}{1 - \bar{c}_j z} \right) \right].$$

4. Proof of Theorem 2. As we shall see in the next section, Theorem 1 follows directly from Theorem 2, whose proof we will concern ourselves with here.

Lemma 2. *Using the notations and definitions of Lemma 1, a function F analytic in D , with values $F^{(j)}(a_i) = B_i^{(j)}$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 1$, will minimize $\|F\|_D$ for all such functions if and only if the function*

$$(3) \quad F_1(z) = \left(\frac{1 - \bar{a}_1 z}{z - a_1} \right) \left[\frac{\lambda F(z) - \lambda B_1^{(0)}}{1 - \lambda^2 \bar{B}_1^{(0)} F(z)} \right],$$

where $\lambda = 1/\|F\|_D$, has the property that if $G(z)$ is analytic in D , and $G^{(j)}(a_i) = F_1^{(j)}(a_i)$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 2$ if $i = 1$ and $j = 0, 1, \dots, l_i - 1$ if $i > 1$, then $\|F_1\|_D \leq \|G\|_D$.

Proof. Suppose $F^{(j)}(a_i) = B_i^{(j)}$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 1$ and that $\|F\|_D$ is a minimum. Let F_1 be as above. Clearly F_1 is analytic in D and $\|F_1\|_D = 1$. If for some other function G analytic in D , with $G^{(j)}(a_i) = F_1^{(j)}(a_i)$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 2$ if $i = 1$ and $j = 0, 1, \dots, l_i - 1$ for $i > 1$, we have $\|G\|_D < \|F_1\|_D$. Then let

$$F^*(z) = \frac{1}{\lambda} \left[\left(\frac{z - a_1}{1 - \bar{a}_1 z} \right) G(z) + \lambda B_1^{(0)} \right] / \left[1 + \lambda \bar{B}_1^{(0)} \left(\frac{z - a_1}{1 - \bar{a}_1 z} \right) G(z) \right].$$

F^* is analytic in D and $\|F^*\|_D < \|F\|_D$. However,

$$\begin{aligned} F(z) - F^*(z) &= \frac{1}{\lambda} \frac{(1 + |\lambda B_1^{(0)}|^2)((z - a_1)/(1 - \bar{a}_1 z))[F_1(z) - G(z)]}{[1 + \lambda \bar{B}_1^{(0)}((z - a_1)/(1 - \bar{a}_1 z))F_1(z)][1 + \lambda \bar{B}_1^{(0)}((z - a_1)/(1 - \bar{a}_1 z))G(z)]} \\ &= \phi(z) \prod_{i=1}^m (z - a_i)^{l_i}, \end{aligned}$$

where $\phi(z)$ is analytic in D . Thus,

$$F^{*(j)}(a_i) = B_i^{(j)} \quad \text{for } i = 1, 2, \dots, m \text{ and } j = 0, 1, \dots, l_i - 1,$$

thus contradicting our hypothesis about F .

In order to prove the "sufficient" portion, let F_1 be as defined in the statement of our lemma and let F_1 have the desired minimizing property. Now suppose H is a function analytic in D with values $H^{(j)}(a_i) = B_i^{(j)}$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 1$, and such that $\|H\|_D < \|F\|_D$. Let

$$H_1(z) = \left(\frac{1 - \bar{a}_1 z}{z - a_1} \right) \left(\frac{\lambda H(z) - \lambda B_1^{(0)}}{1 - \lambda^2 \bar{B}_1^{(0)} H(z)} \right),$$

where $\lambda = 1/\|F\|_D$. The function H_1 is thus analytic in D . $\|H_1\|_D < \|F_1\|_D$ and

$$F_1(z) - H_1(z) = \left(\frac{1 - \bar{a}_1 z}{z - a_1} \right) \left[\frac{(1 - |\lambda B_1^{(0)}|^2)(F(z) - H(z))}{(1 - \lambda^2 \bar{B}_1^{(0)} F(z))(1 - \lambda^2 \bar{B}_1^{(0)} H(z))} \right].$$

Thus we see that H_1 will take on the values $H_1^{(j)}(a_i) = F_1^{(j)}(a_i)$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_1 - 2$ if $i = 1$ and $j = 0, 1, \dots, l_i - 1$ if $i > 1$. This contradicts our assumption about F_1 , and so our lemma follows.

We will now proceed with the proof of Theorem 2.

Proof of Theorem 2. Let F be analytic in D with values $F^{(j)}(a_i) = f^{(j)}(a_i)$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 1$, and suppose $\|F\|_D$ is a minimum for all such functions.

Let us first rename the a_i 's as follows: let $b_k = a_i$ for $\sum_{j=1}^{m-i-1} l_j < k \leq \sum_{j=1}^m l_j$. Then define a finite sequence of functions analytic in D by

$$F_0(z) = \lambda F(z) \quad \text{where } \lambda = 1/\|F\|_D,$$

$$F_k(z) = \left(\frac{1 - \bar{b}_k z}{z - b_k} \right) \left[\frac{F_{k-1}(z) - F_{k-1}(b_k)}{1 - \overline{F_{k-1}(b_k)} F_{k-1}(z)} \right],$$

for $k = 1, 2, \dots, M$ where $M = (\sum_{i=1}^m l_i) - 1$. We then have that $\|F_k\|_D = 1$ for $k = 0, 1, \dots, M$; and furthermore, if G is analytic in D and of the form

$$G(z) = F_k(z) + h(z) \prod_{j=k+1}^{j=M+1} (z - b_j),$$

where h is regular in D , then $\|F_k\|_D \leq \|G\|_D$. This follows from Lemma 2. In particular, $\|F_M\|_D$ is minimal for all functions G analytic in D which satisfy the single condition $G(b_{M+1}) = F_M(b_{M+1})$. It thus follows as a consequence of the maximum principal that F_M is identically this constant. Furthermore, by our assumptions we have that

$$(4) \quad |F_M(b_{M+1})| = 1.$$

Looking back we find that $F_M(b_{M+1})$ can be expressed as a rational function in λ whose coefficients are determined by the a_i 's and the values $f^{(j)}(a_i)$, $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 1$. From (4) we thus obtain a polynomial Λ which we can explicitly determine, and of which λ is a root, i.e. $\Lambda(\lambda) = 0$. The inverse of (2) yields

$$(5) \quad F_{k-1}(z) = \left[\frac{((z - \bar{b}_k)/(1 - \bar{b}_k z))F_k(z) + F_{k-1}(b_k)}{1 + \overline{F_{k-1}(b_k)}((z - \bar{b}_k)/(1 - \bar{b}_k z))F_k(z)} \right],$$

and allows us to calculate F_0 and then F once we have determined λ . We note that the values $F_{k-1}(b_k)$, $k = 1, 2, \dots, M$, are functions of λ , and that as defined by (5), each function F_{k-1} will be analytic in D with $\|F_{k-1}\|_D = 1$ (as desired) provided that the function F_k is and that $|F_{k-1}(b_k)| < 1$. Thus λ will be the largest

positive root of the polynomial Λ which satisfies the condition $|F_{k-1}(b_k)| < 1$, for $k = 1, 2, \dots, M$. We shall give an example of how Λ , λ and thus F can be determined later.

From (5) it is also evident that the function F_{k-1} will be rational and $\|F_{k-1}\|_D = 1$ for $k = 1, 2, \dots, M$, since $F_M(z) \equiv \pm 1$. In particular the function F will have the form

$$F(z) = \frac{1}{\lambda} \prod_{j=1}^K \left(\frac{z - c_j}{1 - \bar{c}_j z} \right),$$

where $|c_j| < 1$ for $j = 1, 2, \dots, K$ and $K \leq M = (\sum_{i=1}^M l_i) - 1$. Note that due to possible cancellation of terms we might have $K < M$.

The uniqueness of F now follows easily. Suppose G is another function analytic in D which assumes the values $G^{(j)}(a_i) = F^{(j)}(a_i)$ for $i = 1, 2, \dots, m$ and $j = 0, 1, \dots, l_i - 1$, and such that $\|G\|_D = \|F\|_D$. Let λ be as before and define a sequence $\{G_k\}_{k=1}^M$ from G as we did from F . We will then conclude that $|G_M(b_{M+1})| = 1$, and furthermore, we will find that $G_M(b_{M+1})$ can be expressed as the same rational function of λ that we obtained for $F_M(b_{M+1})$, namely one whose coefficients depend only on the prescribed values of F at the points $\{a_i\}_{i=1}^m$. As a result we find that $G_M(b_{M+1}) = F_M(b_{M+1})$, and then working backwards we obtain $G_0 \equiv F_0$, and thus $G \equiv F$.

Proof of Theorem 1. Let g be as in Corollary 1, the unique analytic function of "best approximation" to f on D . We may then write

$$\begin{aligned} \|f - g\|_U &= \left\| f_1(z) - g(z) \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} \right\|_D \\ (6) \quad &\leq \left\| f_1(z) - p_n(f, U)(z) \prod_{i=1}^m \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} \right\|_D \\ &= \|f - p_n(f, U)\|_D, \quad \text{for } n = 0, 1, 2, \dots \end{aligned}$$

Furthermore, the function g is not only analytic in D but continuous in $\bar{D} = D + U$, and so by Walsh's theorem [4, p.98], given any $\varepsilon > 0$, there exists a polynomial q such that $\|g - q\|_U < \varepsilon$. Thus for n sufficiently large (greater than the degree of q), we have

$$\|f - p_n(f, U)\|_U \leq \|f - q\|_U \leq \|f - g\|_U + \varepsilon,$$

which together with (6) implies that

$$(7) \quad \lim_{n \rightarrow \infty} \|f - p_n(f, U)\|_U = \|f - g\|_U.$$

From (7) and the uniqueness of g it now follows that every subsequence of $\{p_n(f, U)\}_{n=0}^{\infty}$ contains itself, a subsequence that converges uniformly on compact subsets of D to g . However, the sequence $\{p_n(f, U)\}_{n=0}^{\infty}$ is equicontinuous on every compact subset of D , and so our theorem follows.

5. **An example.** We now give a simple application of Theorem 1. Let $f(z) = 1/[z(z - 1/2)]$. In order to find $\lim_{n \rightarrow \infty} [f - p_n(f, U)]$ we first find that function F analytic in D , with values:

$$F(0) = f_1(0) = 1,$$

$$F(1/2) = f_1(1/2) = 4/3, \text{ for which } \|F\|_D \text{ is a minimum, where}$$

$$f_1(z) = 1/(1 - z/2).$$

Let $\lambda = 1/\|F\|_D$, $F_0 = \lambda F$, and $F_1(z) = (1/z)[(F_0(z) - \lambda)/(1 - \lambda F_0(z))]$. The function F_1 must then be a constant, and

$$\left| F_1\left(\frac{1}{2}\right) \right| = \left| 2 \left[\frac{\lambda((4/3) - 1)}{1 - \lambda^2(4/3)} \right] \right| = 1.$$

In order to determine λ we must then find the largest positive root of the quadratic equation $2\lambda/[3(1 - \lambda^2(4/3))] = \pm 1$ which satisfies $|F_0(0)| < 1$ or $\lambda < 1$. This yields $\lambda = ((13)^{1/2} - 1)/4 = .65 \dots$. Thus $F_1(z) \equiv F_1(1/2) = 2\lambda/(3 - 4\lambda^2) = 1$, and $F(z) = (1/\lambda)[(z - \lambda)/(1 - \lambda z)]$. It then follows that for $z \in D$,

$$\lim_{n \rightarrow \infty} [f(z) - p_n(f, U)(z)] \left[\frac{z(z - 1/2)}{(1 - z/2)} \right] = \frac{1}{\lambda} \left(\frac{z - \lambda}{1 - \lambda z} \right),$$

and so

$$\lim_{n \rightarrow \infty} \|f - p_n(f, U)\|_U = \frac{1}{\lambda} = 1.53 \dots$$

6. **The general case.** We now consider the case where f is a function that is continuous on a closed Jordan curve Γ and meromorphic inside Γ .

Letting Ω denote the interior of Γ we have, by Riemann's mapping theorem, that there exists a function $w = \phi(z)$ that is regular in Ω and maps Ω conformally onto the open unit disc D . Furthermore, since the boundary of Ω is Γ , a closed Jordan curve, ϕ may be extended continuously to Γ so that it will map Γ onto the unit circle U in a one-to-one manner [3, Vol. III, p. 70].

The function $f \circ \phi^{-1}$ will then be meromorphic in D and continuous on U . If we let $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ be the poles of f in Ω , then $\{a_1, a_2, \dots, a_m\}$ where $a_i = \phi(\alpha_i)$ for $i = 1, 2, \dots, m$ will be the poles of $f \circ \phi^{-1}$ in D . Applying Theorem 1 to $f \circ \phi^{-1}$ on U we have

$$(8) \quad \lim_{n \rightarrow \infty} [f \circ \phi^{-1}(w) - p_n(f \circ \phi^{-1}, U)(w)] \prod_{i=1}^m \left(\frac{w - a_i}{1 - \bar{a}_i w} \right)^{l_i} \\ = \frac{1}{\lambda} \prod_{j=1}^K \left(\frac{w - c_j}{1 - \bar{c}_j w} \right) \quad \text{for } |w| < 1,$$

where λ , K and the c_j 's, $j = 1, 2, \dots, K$, are as described in Theorem 1, and the convergence is uniform on compact subsets of D .

Theorem 3. *Using the above notation,*

$$\begin{aligned} \lim_{n \rightarrow \infty} [f(z) - p_n(f, \Gamma)(z)] \prod_{i=1}^m \left(\frac{\phi(z) - a_i}{1 - \bar{a}_i \phi(z)} \right)^{l_i} \\ = \frac{1}{\lambda} \prod_{j=1}^K \left(\frac{\phi(z) - c_j}{1 - \bar{c}_j \phi(z)} \right) \quad \text{for } z \in \Omega, \end{aligned}$$

where the convergence is uniform on compact subsets of Ω .

Proof. Let $P_n = p_n(f \circ \phi^{-1}, U)$, $Q_n = p_n(f, \Gamma)$ for $n = 0, 1, \dots$, and $R(w) = \frac{1}{\lambda} \prod_{j=1}^K (w - c_j)/(1 - \bar{c}_j w)$.

If $z \in \Omega$ and $w = \phi(z)$ we can write

$$\begin{aligned} (9) \quad & \left| R \circ \phi(z) - [f(z) - Q_n(z)] \prod_{i=1}^K \left(\frac{\phi(z) - a_i}{1 - \bar{a}_i \phi(z)} \right)^{l_i} \right| \\ & \leq \left| R(w) - [f \circ \phi^{-1}(w) - P_n(w)] \prod_{j=1}^K \left(\frac{w - a_j}{1 - \bar{a}_j w} \right) \right| \\ & \quad + |P_n(w) - Q_n \circ \phi^{-1}(w)|. \end{aligned}$$

Consequently, our theorem will follow if it is shown that the sequence $\{P_n - Q_n \circ \phi^{-1}\}_{n=0}^{\infty}$ converges to zero uniformly on compact subsets of D . In order to accomplish this we first prove that

$$(10) \quad \lim_{n \rightarrow \infty} \|f - Q_n\|_{\Gamma} = \lim_{n \rightarrow \infty} \|f \circ \phi^{-1} - P_n\|_U.$$

To this end, we observe that since ϕ^{-1} is regular in D and continuous in $D + U$, it can be arbitrarily approximated by polynomials on U . This follows from Walsh's theorem [4, p. 98]. Thus given any $\varepsilon_1 > 0$ there exists a polynomial S_m such that $\|\phi^{-1} - S_m\|_U < \varepsilon_1$, and given any $\varepsilon_2 > 0$, by choosing ε_1 sufficiently small we will have

$$\begin{aligned} (11) \quad & \|f \circ \phi^{-1} - Q_n \circ S_m\|_U \leq \|f \circ \phi^{-1} - Q_n \circ \phi^{-1}\|_U + \varepsilon_2 \\ & = \|f - Q_n\|_{\Gamma} + \varepsilon_2. \end{aligned}$$

But $Q_n \circ S_m$ is a polynomial of degree nm , so we have

$$\|f \circ \phi^{-1} - P_{nm}\|_U \leq \|f \circ \phi^{-1} - Q_n \circ S_m\|_U,$$

which then implies that

$$\lim_{n \rightarrow \infty} \|f \circ \phi^{-1} - P_n\|_U \leq \lim_{n \rightarrow \infty} \|f - Q_n\|_{\Gamma}.$$

Similarly given any $\varepsilon_3 > 0$ there exists a polynomial r_k such that $\|\phi - r_k\|_{\Gamma} < \varepsilon_3$, and given any $\varepsilon_4 > 0$, by choosing ε_3 sufficiently small we will have

$$\|f - P_n \circ r_k\|_{\Gamma} \leq \|f - P_n \circ \phi\|_{\Gamma} + \varepsilon_4 = \|f \circ \phi^{-1} - P_n\|_U + \varepsilon_4.$$

Repeating our previous argument we obtain (10).

Now let us suppose that on some nonempty compact subset of D , $\{P_n - Q_n \circ \phi^{-1}\}_{n=0}^{\infty}$ does not uniformly converge to zero. Then since the sequences $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n \circ \phi^{-1}\}_{n=0}^{\infty}$ are equicontinuous on each compact subset of D , there exists a subsequence $\{P_{k_n} - Q_{k_n} \circ \phi^{-1}\}_{n=0}^{\infty}$, such that:

(i) $\{P_{k_n}\}_{n=0}^{\infty}$ converges uniformly on compact sets of D to a function L_1 analytic in D ,

(ii) $\{Q_{k_n} \circ \phi^{-1}\}_{n=0}^{\infty}$ converges uniformly on compact sets of D to a function L_2 analytic in D , and

(iii)

$$(12) \quad \|L_1 - L_2\|_D > 0.$$

By (10) we have

$$(13) \quad \begin{aligned} & \left\| [f \circ \phi^{-1}(w) - L_1(w)] \prod_{i=1}^m \left(\frac{w - a_i}{1 - \bar{a}_i w} \right)^{l_i} \right\|_D \\ &= \left\| [f(z) - L_2 \circ \phi(z)] \prod_{i=1}^m \left(\frac{\phi(z) - a_i}{1 - \bar{a}_i \phi(z)} \right)^{l_i} \right\|_{\Omega} \\ &= \left\| [f \circ \phi^{-1}(w) - L_2(w)] \prod_{i=1}^m \left(\frac{w - a_i}{1 - \bar{a}_i w} \right)^{l_i} \right\|_D. \end{aligned}$$

Now by (8),

$$L_1(w) = f \circ \phi^{-1}(w) - R(w) \prod_{i=1}^m \left(\frac{1 - \bar{a}_i w}{w - a_i} \right)^{l_i}.$$

Furthermore, by Theorem 1, $L_1(w)$ is the unique "analytic function of best approximation" to $f \circ \phi^{-1}$ on D . This uniqueness together with (13) implies $L_1 \equiv L_2$ and thus our theorem follows.

7. Approximation to arbitrary continuous functions. Although Theorem 3 is not in general applicable for an arbitrary continuous function f on a closed Jordan curve Γ (since f is not in general defined in the interior of Γ), a somewhat weaker version can be obtained.

Let Γ and ϕ be as in §6. For a function f continuous on Γ let

$$\rho(f, \Gamma) = \lim_{n \rightarrow \infty} \|f - p_n(f, \Gamma)\|_{\Gamma}.$$

Theorem 4. *Let f be continuous on Γ ; then given any $\varepsilon > 0$ there exists a rational function R of the form*

$$R(w) = \frac{1}{\lambda} \left[\prod_{i=1}^m \left(\frac{1 - \bar{a}_i w}{w - a_i} \right)^{l_i} \right] \left[\prod_{j=1}^K \left(\frac{w - c_j}{1 - \bar{c}_j w} \right) \right],$$

such that: (i) $|\rho(f, \Gamma) - \|R \circ \phi\|_\Gamma| < \varepsilon$, and

(ii) there exists a polynomial q such that $\|(f - q) - R \circ \phi\|_\Gamma < \varepsilon$.

Proof. If $\rho(f, \Gamma) = 0$ the theorem is true if we let $R \equiv 0$. Thus we may assume $\rho(f, \Gamma) > 0$. It is well known [3, Vol. III, p.100] that if f is continuous on a closed Jordan curve Γ , then there exists a sequence of rational functions that converge to f uniformly on Γ . In particular, there exists a rational function V for which $\|f - V\|_\Gamma < \varepsilon/2$. Moreover, V is of the form $V(z) = \sigma(z) / \prod_{i=1}^m (z - \alpha_i)^{l_i}$, where $\sigma(z)$ is regular on Γ and in its interior and the α_i 's lie in the interior of Γ . Since we have assumed that $\rho(f, \Gamma) > 0$ it will then follow that V has at least one pole inside Γ if ε is sufficiently small. Let us thus assume that this is the case.

Applying Theorem 3 to V on Γ , let $R^*(z)$ be the limit of the sequence

$$\left\{ [V(z) - p_n(V, \Gamma)(z)] \prod_{i=1}^m \left(\frac{\phi(z) - a_i}{1 - \bar{a}_i \phi(z)} \right)^{l_i} \right\}_{n=0}^{\infty}$$

for z in the interior of Γ , where ϕ is as before and $a_i = \phi(\alpha_i)$ for $i = 1, 2, \dots, m$. The function $R^*(z)$ is then of the form

$$R^*(z) = R \circ \phi(z) \prod_{i=1}^m \left(\frac{\phi(z) - a_i}{1 - \bar{a}_i \phi(z)} \right)^{l_i},$$

where

$$R(w) = \frac{1}{\lambda} \left[\prod_{i=1}^m \left(\frac{1 - \bar{a}_i w}{w - a_i} \right)^{l_i} \right] \left[\prod_{j=1}^K \left(\frac{w - c_j}{1 - \bar{c}_j w} \right) \right].$$

Now for any polynomial p and for $z \in \Gamma$,

$$\begin{aligned} |V(z) - p(z)| &\leq |V(z) - f(z)| + |f(z) - p(z)| \\ &\leq \varepsilon/2 + |f(z) - p(z)|. \end{aligned}$$

Similarly,

$$|f(z) - p(z)| \leq \varepsilon/2 + |V(z) - p(z)|.$$

Thus $|\rho(V, \Gamma) - \rho(f, \Gamma)| < \varepsilon$. But $\rho(V, \Gamma) = \|R \circ \phi\|_\Gamma$ so (i) follows.

The function $(V - R \circ \phi)$ is continuous on Γ and regular interior to Γ so again by [4] there exists a polynomial q such that $\|V - (R \circ \phi) - q\|_\Gamma < \varepsilon/2$. Thus,

$$\begin{aligned} \|(f - q) - (R \circ \phi)\|_\Gamma &\leq \|f - V\|_\Gamma + \|V - (R \circ \phi) - q\|_\Gamma \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

and so (ii) follows.

Remark. If f and Γ are as in Theorem 4, and if one can find a sequence $\{R_n\}_{n=0}^\infty$ of rational functions converging to f uniformly on Γ , and if one can calculate the mapping function ϕ for the interior of Γ , then it is now possible to determine a sequence of polynomials $\{q_k\}_{k=0}^\infty$ such that $\lim_{k \rightarrow \infty} \|f - q_k\|_\Gamma = \rho(f, \Gamma)$.

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